Analytic Hierarchy Process *(*Draft 2008-04-30*)*

[r.r]

Introduction

The purpose of this manual is to introduce and explain how to use a decision support framework and associated tools called the Analytic Hierarchy Process (AHP). This approach will help you to both structure and analyze multilevel decision problems that would normally defeat an intuitive solution. This is an alternative approach to asking people to intuitively assign numbers to variables and then use those numbers to rank additional variables. AHP improves on this by making the judgments transparent as well as providing a measure of *judgemental consistency*, a very valuable feature.

My presentation level in this manual is aimed at the student and beginning researcher in the fields of management and marketing, as well as workers in science and engineering. The applications of AHP are extremely varied and include insight into problems like the following: setting priorities, generating options, choosing alternatives, getting and prioritizing requirements, allocating resources, risk assessment, predicting outcomes, measuring performance, system design and stability, optimizing, planning, conflict resolution, and cost-benefit analyses.

The Analytic Hierarchy Process (AHP) was devised by Thomas Saaty in the early 1970's to help in prioritizing very complex decision alternatives involving multiple stakeholders and multiple goals. His method guides structuring, measuring, prioritizing, as well as consistency checking of the judgements bearing on the prioritization. The structuring part uses the idea that humans automatically group like experiences into clusters, and, if complex enough, construct hierarchies of layers of those clusters. If asked to measure the impact of the bottom layer of clusters (such as 'action' alternatives), on the overall, topmost goal, (such as 'overall benefit') Saaty's method allows a systematic roll-up of impacts of child layers on parent layers until the top layer is reached. The result is a prioritization of the bottom layer in terms of its impact on the top layer. That is, in common situations, this answers the question of "which alternative should I choose to achieve a stated top level goal". The beauty of the method is that it allows the same question to be answered methodically, even if there are multiple layers of judgements between the alternatives/choices and the top goal.

Ok, let's get started with a simplified example:

Example 1 Hiring a Business Analyst

Just to introduce the AHP ideas, suppose I want to hire a business analyst and have three candidates in mind. Since the job is complex, I want to judge the candidates on several different criteria. The decision hierarchy graph is shown below. The full AHP method includes generating the numbers like those shown on the graph but just suppose I already have gotten these somehow. I want to illustrate how you can 'roll up' decisions from the bottom up thru intermediate layers so that the alternatives are ranked in terms of the top most goal, no matter how many intermediate layers. Think of the numbers on the lines as "ratio weights" or importance/intensity ratio factors. Getting these numbers is a major task of AHP and we will go into that shortly, but I want to concentrate here on how you can 'roll-up' decisions through multiple layers, in this case two layers, given you *have* the ratio weights.

O.K., I want to hire a business analyst, so, first I set up the top level goal, *Hire a Business Analyst* and two sub-goals: desired *People skills* and desired *Technical skills*. I go ahead and weight the relative importance of people skills to technical skills, say, in the *ratio* of 4/2 (this is irrespective of any candidates yet involved). That is, I am thinking that people skills are twice as important as technical skills, for this particular position. Writing these ratios so that these weights add to “1”, yields 4/6 and 2/6. This specifies the normalized relative ratio of importance of *People skills* to *Technical skills*, in the context of the overall *Hire a Business Analyst.*

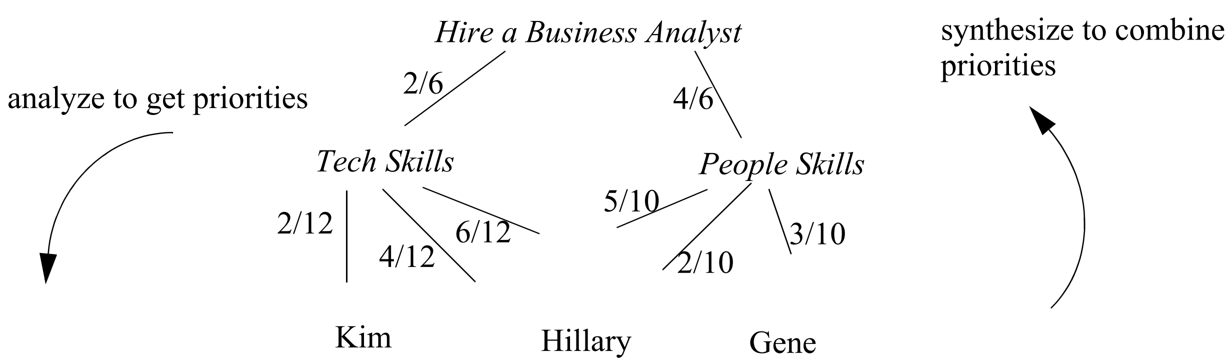
Now I move down a level and assess the three candidates *within* each of the sub-goals, *People skills* and *Technical skills*. So, given the context of *People skills*, suppose that my judgements specify that Hillary is twice as good as Kim and that Gene is three times as good as Kim. My weightings also say that Gene is 6/4 times as good as Hillary. These judgements are in the context of the *People skills*. (Remember, I haven’t yet shown you how to get these numbers, but just accept them for right now to see how this phase works).

Now I move to the other sub-goal and assess these people within that context, *Technical skills*: Suppose that my resultant judgements show Kim as 5/2 as good as Hillary and 5/3 as good as Gene. I have also determined that Gene is 3/2 as good as Hillary.

So, who’s best qualified? This is a typical decision problem where in one context, person A is better, while in another context person B is superior. How to synthesize multiple levels of preferences is the questions here. The AHP approach takes all this into account and rolls up through each level to suggest a final ranking/prioritization of the bottom most alternatives, within the top most goal. In this case, the AHP will suggest a ranking of the three candidates, within the top goal, *Hire a Business Analyst*.

Here’s how the roll-up works.

- Take the vector of candidates priorities relative to *Technical skills*, {2/12, 4/12, 6/12} and weight each component of that vector by 2/6, which is the weight of Technical skills relative to the top most goal, *Hire a Business Analyst*. The result is {4/72, 8/72, 12/72}. This is the priority/importance/impact of each candidate on the top most goal, relative to their *Technical skills*.
- Take the vector of candidate priorities relative to *People skills*, {5/10. 2/10, 3/10} and weight that by 4/6 which is the weight of People skills relative to the top most goal. This yields {20/60, 8/60, 12/60}. This is the priority/importance/impact of each candidate on the top most goal, relative to their *People skills*.
- Combine the two priority vectors to give {4/72+ 5/10, 4/12} +{2/10, 6/12 + 12/72} = {7/18, 11/45, 11/30} = {0.39, 0.24, 0.37}
- That vector, {0.39, 0.24, 0.37} represents the final priorities of the candidates relative to the top most goal. This represents a ‘roll-up’ through the intermediate layer of Technical skills and People skills.



The overall ranking of Kim, Hillary, Gene in the context of the top most goal:

$$\frac{2}{6} * \begin{pmatrix} 2/12 \\ 4/12 \\ 6/12 \end{pmatrix} + \frac{4}{6} * \begin{pmatrix} 5/10 \\ 2/10 \\ 3/10 \end{pmatrix} = \begin{pmatrix} 7/18 \\ 11/45 \\ 11/30 \end{pmatrix} = \begin{pmatrix} 0.39 \\ 0.24 \\ 0.37 \end{pmatrix}$$

FIGURE 1. Illustration of the Synthesis Stage of AHP

The result of my synthesis is a final priority vector, {0.39, 0.24, 0.37} which gives the relative merits of the three candidates Kim, Hillary, and Gene. This says that Kim is 0.39/0.24 more preferable than Hillary, and Kim is 0.39/0.37 more preferable than Gene. Based on these priorities, Kim and Gene are really too close to differentiate and additional criteria could be used, such as formal degrees obtained or performance on a pop quiz!

This conclusion is very useful but of course there is the issue of how to get these weights in the first place. That is the analysis phase of AHP and involves paired comparisons as I'll discuss shortly. First, though, I'll give a few more sketches of the kinds of problems you can formulate and solve using these ideas.

*Note: If you can't wait to see how these numbers might be determined, jump to "Is AHP Reasonable? Checking the Method Against an Absolute Scale" on page 6 and subsequent examples and calculations.

Examples of the AHP

I would like to show you some of the *structures* that can be analyzed using AHP. I won't do the numbers here, but just show you how you might go about setting up the structure that will hold the questions to be asked and then answered.

Think of the lines joining the elements of the diagram as ultimately carrying ratios representing the importance of that element relative to its parent element, that is, the element above it in the hierarchy. In example 2 below, the parent of the *Total Cost (C)* element is the element, *Purchase of a Document Management System*. The line joining them would carry the ratio of the importance of *Total Cost (C)* relative to its parent goal. The other children of the *Purchase of Document Management System* which are *Technical Component* and *Maintenance Service*, would carry ratio weights representing *their* importance in the context of the parent element. The three weights would add up to unity.

Organizational/Business Decisions

Example 2. Capital Equipment/Software Application, with Nested Criteria

Below is a decision hierarchy that has nested criteria within the Tech-Component. I have labeled these as Level 3 Sub-criteria. Here, it is possible to break down the technical criterion into finer components (F1, F2, F3) that can then be analyzed by the standard paired comparison approach I will show you later. I will analyze this hierarchy numerically later in this manual, but for now just, appreciate the potential for nested criteria.

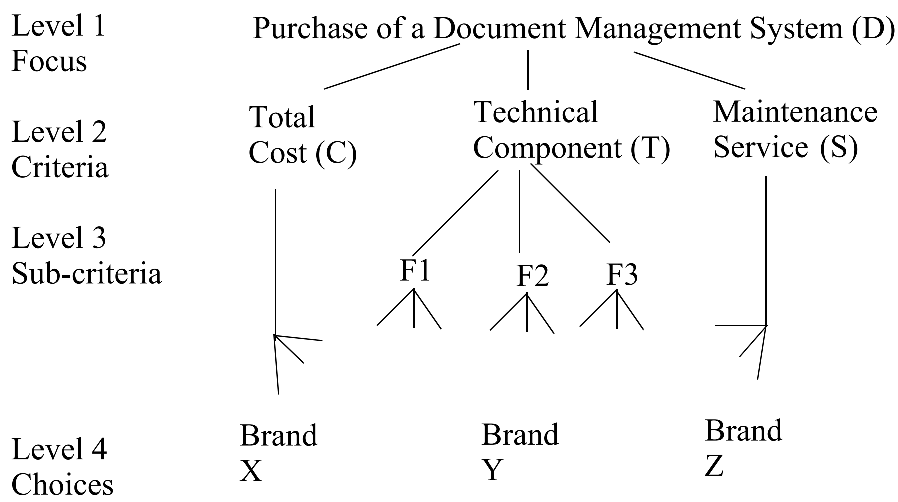


FIGURE 2. Four Level Hierarchy for Capital Acquisition

Example 3. Combining Benefit and Cost Hierarchies for Resource Allocation

It has proven most useful to derive the benefits and the costs of choices by using separate hierarchies rather than considering them simultaneously. (You can do them in one 'tree', but its harder to interpret). This means setting up a *benefit* hierarchy as well as a separate *cost* hierarchy, and then using their outcomes to compute ratios of benefit to cost. Look at the two hierarchies of Figure 3 on page 5.

Suppose I determined that the three employee policies, P1, P2, and P3, had benefits and costs as follows relative to the overall *Corporate Survivability* goal:

benefit priorities {0.6, 0.3, 0.1}

cost priorities of {0.2, 0.3, 0.5},

Then I could calculate my benefit to cost ratios by dividing each component of the benefit vector with the corresponding component of the cost vector. The result is a benefit/cost ratio vector that I can use in various ways, such as allocating resources.

benefit/cost ratios = {3, 1, 0.2} and normalizing this vector, so the components add to "1", yields

{0.714286, 0.238095, 0.047619}

One way to look at this would be to say that 71% of resources should go to exploring/implementing P1 while 24% ought to go to P2, and only 5% ought to go to P3. There is always the possibility of setting up multiple policies, such as exploring/investigating and perhaps synthesizing P1 *and* P2.

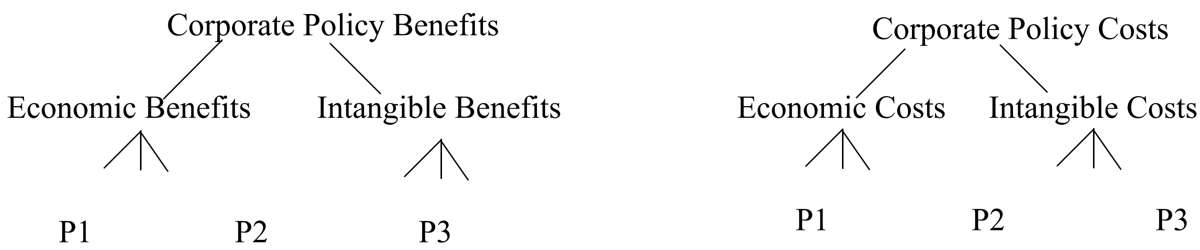


FIGURE 3. Combining Separate Benefit and Cost Hierarchies for Benefit/Cost Ratios

Example 4. Personnel Choices

This next hierarchy is for selection of a faculty candidate.

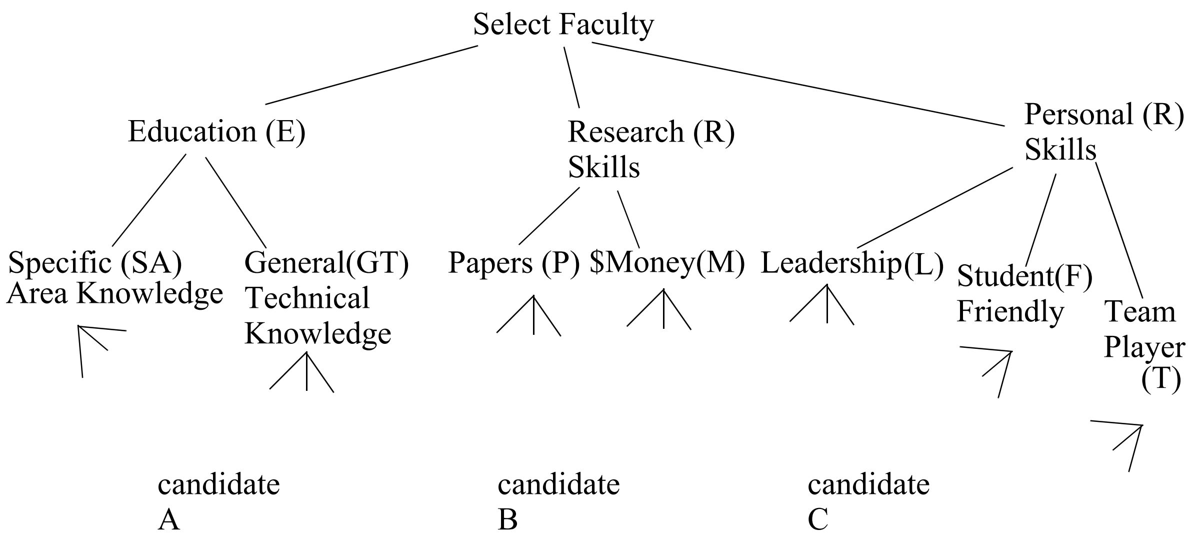


FIGURE 4. Faculty Selection Hierarchy

Main Ideas of the AHP

The central idea of the AHP is: *Use Paired Comparisons to Deduce Intrinsic Priorities*

The overall plan of the AHP process is to do a sequence of pairwise ratio comparisons (each comparison is done within a given goal context), and from these ratios, calculate what the user's priorities must have been. (It is usually easier for a participant to judge two elements at a time rather than try to juggle 3 or more).

Within that basic idea we find the following structures and processes:

- A layered hierarchy of clustered situational features that allow decision to go 'down' the hierarchy during the analysis stage, and "up" the hierarchy during the synthesis stage.
- The measuring part is built into the method since requiring paired comparison judgements automatically generates a ratio scale that can be used in the prioritizing stage. (See "Priority Scale for AHP" on page 12.)

- The individual priorities can be extracted from the judged *paired comparisons* by some standard math, not easy, but standard. What this means is that from a collection of comparisons at a given level, the relative individual element priorities/weights can be deduced/calculated.
- Consistency can be calculated by checking the inter-consistency of judgments against a baseline of tabled random judgments. (See "Random Consistency Index Table" on page 12.)
- The power of the method is that it allows both quantitative measures, such as 'money' or 'cubic feet of concrete', as well as qualitative judgments, such as 'color preference' or 'personal security feelings', to be equal contributors throughout the hierarchy.

Saaty Speaks:

Here is Thomas Saaty, the creator of the AHP method, explaining his method (this is from his book *Decision Making for Leaders*, 1990, hence the reference to 'leaders' in the text below).

> The process contributes to solving complex problems by structuring a hierarchy of criteria, stakeholders, and outcomes and by eliciting judgments to develop priorities. It also leads to prediction of likely outcomes according to these judgments.
>
> The outcome can be used to rank alternatives, allocate resources, conduct benefit/ cost comparisons, exercise control in the system by evaluating the sensitivity of the outcome to changes in judgment, and carry out planning of projected and desired futures. A useful by-product is the measurement of how well the leader understands the relations among factors. Although people generally are not consistent, the main concern here is the strength of their inconsistency. Is their understanding close to capturing the interactions observed? Or is it a random understanding that only hits the target now and then?
>
> . . . Basically, the AHP is a method of breaking down a complex, unstructured situation into its component parts, or variables, into a hierarchic order; assigning numerical values to subjective judgments on the relative importance of each variable, and synthesizing the judgments to determine which variables have the highest priority and should be acted upon to influence the outcome of the situation. (Saaty, (1990) *Decision Making for Leaders, pg 1*)

Is AHP Reasonable? Checking the Method Against an Absolute Scale

(As an aside, I usually bring to class a common bathroom scale that measures pounds, and illustrate AHP as below. The students judge various weight ratio pairs by hand and then check their judgments and consistency against the actual scale determined weights.)

In the scenario below I am going to assume I used an absolute scale, like a bathroom scale, just to *show the pattern* to be used even when there are *no measuring devices* and personal human judgement must be substituted for an absolute device. So, in this discussion *I already know* the absolute weights of the elements being considered and so can calculate exactly the ratios that would normally only be estimated with paired comparisons.

Consider three stacks of books as in the figure below.

Suppose we also have an accurate scale that we can use to weigh each stack. In this situation my goal is to assess the stacks in the context of their *weight*. So *weight* is the overall criterion that I will use to judge the relative ranking of these stacks by. I could have picked another attribute to judge the stacks by, such as price or aesthetic attractiveness, but weight allows an understandable comparison criterion that we can objectively check the consequences of. In this situation we have an objective criterion and a standard scale that shows pounds. (Think about the general case though where there is no scale, just your own or your colleagues personal judgements - that is the general situation we are setting up to get some intuition about).

O.k, let's start. Suppose I set up stack A = 4 books, stack B = 2 books, and stack C = 1 book.

Now I *actually* weigh each stack and suppose I get A= 4 lb., B= 2 lb. and C= 1 lb.

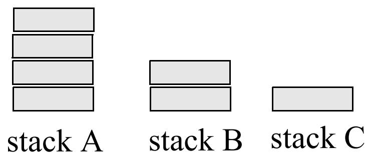


FIGURE 5. Verifying the AHP Method with Known Weights and Hence, Known Ratios

Of course, if I compare the ratio weight of stack A to the weight of stack A, I get 4 to 4, or 4/4 = 1. As will become clear as we go along, establishing this ratio of weights (or importance ratios) is the key to the method and is the hard part when you don't have a standard scale handy! Similarly the ratio of A/B is 4/2 and ratio of A/c would be 4/1.

> You already know there are no absolute measures for values like beauty or congeniality, but even so, we often compare one person to others on this basis. Or, think of city planners trying to judge the relative merits of two or more memorial building proposals on multiple criteria such as cost, aesthetics, public acceptance, traffic disruption, pollution, and many more. The AHP supports these kinds of judgments by means of paired comparisons and a little math, so read on!

Note that I will reasonably assume that if I know the ratio of A/B then I automatically know the ratio of B/A. For example, I know the weight ratio of A/B = 4/2. From this I can automatically fill in the table entry for B/A, which would be 2/4. For the table below, I have indicated, by an asterisk the entries that were automatically inserted. Additionally, I have shown the symbolic ratios that the number represent, where w1 is the weight of stack A, w2 is the weight of stack B, and w3 is the weight of stack C.

TABLE 1. Weight Ratios of Stacks of Books - A Table of Paired Comparisons

WEIGHT (Comparison Context)	stack A	stack B	stack C
stack A	1/1* = w1/w1	4/2 = w1/w2	4/1 = w1/w3
stack B	2/4* = w2/w1	1/1* = w2/w2	2/1 = w2/w3
stack C	1/4* = w3/w1	1/2* = w3/w2	1/1* = w3/w3

The Matrix Representation

To do the math to find priority vectors, that is, the vector of relative weights of the stacks, as well as the consistency index, it is convenient to think of the table above as a *matrix* of paired comparisons. You wouldn't have to do that and could actually use a tool like Excel to make the needed calculations from a table, but I like the math perspective and so will represent the table as a matrix.

To repeat, from the *pair ratios*, can we get back the ***relative*** *individual weights* of each stack. Since we know everything already, we ought to find that stack A is 2 times as 'weighty' as stack B, and 4 times as 'weighty' as stack C. Similarly, we should find that stack B is 2 times the weight of stack C. In this case, we could inspect the matrix to see that, but in practice, we don't have a standard scale, and so we get approximations to these ratios. What follows is a math way to extract those individual relative 'weights' from the collected pairs. These relative rankings are called *priorities* in the AHP language.

O.k, I have taken the table and recast it as a matrix as below in Figure 6 on page 8. The method to actually calculate the relative priorities, given the matrix (or the table if you prefer), follows this figure.

Matrix WR (the weight ratios)

$$\begin{pmatrix} 1 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}$$

Sum the columns

$$\left\{\frac{7}{4}, \frac{7}{2}, 7\right\}$$

Divide each element in the column by its column sum to yield:
Matrix N (the columns now sum to 1, and the columns are called 'normalized')

$$\begin{pmatrix} \frac{4}{7} & \frac{4}{7} & \frac{4}{7} \\ \frac{2}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

Sum the rows of matrix N to yield:

$$\begin{bmatrix} 12/7 \\ 6/7 \\ 3/7 \end{bmatrix}$$

Divide by the number of columns (3 in this case)

$$\begin{bmatrix} 4/7 \\ 2/7 \\ 1/7 \end{bmatrix}$$

Done! These are the relative weights (priorities), this is also the 'eigenvector'

FIGURE 6. The Weight Ratios Matrix: Calculating Priorities from the Paired Comparisons

The last vector we calculated is the normalized (the elements sum to 1.0) vector of priorities. You can see that the relative ranks are exposed. For example, A/B = 4/7 /2/7 = 2, while A/C = 4/7 / 1/7 = 4

This was our objective, to get the individual relative weights, just by knowing the pair ratios.

In this perfect case, you can see that knowing the paired comparisons allowed us to deduce the exact relative individual rankings. In the general case, nothing is so obvious, but the method still works.

Deviation From Consistency, the Consistency Index (CI) and the Consistency Ratio (CR)

If we do a little more work, we can determine how *inconsistent* our judgments are. (The AHP is the best (only?) method I know of that will allow such calculations. This is crucially important since it allows me to review/revise inconsistent judgments).

The deviation from consistency is calculated as a Consistency Index (CI):

Consistency Index (CI) = (eigenvalue - n)/(n-1)

"n" is the number of columns (or rows) of the paired comparison matrix

In the case of the 'weight' matrix, above, this is a perfect case, and I get CI= (3-3)/(3-1) = 0, perfect!

Generally we go further and calculate an additional parameter, the Consistency Ratio (CR).

Consistency Ratio = CI/(Random CI)

This ratio will be used to judge a matrix's CI against a random matrix's CI. The rule of thumb is that your CI should be *less that 10%* of the *Random CI*.

The Random CI is a simulated value based on random reciprocal matrix CIs. There is a table of these to use for various sizes of matrices. For a 3x3 matrix, the Random CI = 0.58 (See "Random Consistency Index Table" on page 12)

Manual Procedure to Calculate the Priority Vector and Eigenvalue

[Note: for you math wizards out there, we are finding the maximum eigenvalue and its' associated eigenvector. The eigenvector will be called the *priority vector* in the AHP vocabulary and the maximum *eigenvalue* will be used in the check for consistency].

Start with your matrix/table of paired comparisons: refer to Figure 6 on page 8, shown above, for the example matrix used to illustrate these calculations.

1. Total up each column and divide each element in the that column by that total.
   For example, the first column total of matrix P, shown in Figure 2 below, is 7/4, and dividing each element of the first column by that total gives column 1 of matrix N.
2. Total up each row, and divide by the number of columns of the matrix.
   For example, row 1 of matrix N totals 12/7. Dividing this by 3, gives 4/7. For row 2 the total is 6/7 and final value is 2/7. For row 3 the final value is 3/7 divided by 3 = 1/7

Notice that the final vector, 4/7, 2/7, 1/7 is *normalized*, that is, it sums to one. Further, you can read off the priorities just by looking at the ratios. For example 4/7 divided by 2/7 yields "2", which is the relative weight of stack A to stack B. Similarly, 2/7 divided by 1/7 yields a relative weight ratio of "2" between stack B and stack C. We have now found the priority vector which is the same as the 'principle eigenvector'.

3. To find the *eigenvalue*, matrix multiply the comparison matrix by the priority vector found in step 2. Call this resultant vector "V". (This vector is a multiple of the priority vector, and we are looking for that multiple, that's the eigenvalue.)
4. Divide "V" by the priority vector, element by element. Average these elements and that is the best estimate of the eigenvalue. (See Figure 7 on page 10)

Consistency Comparisons

To see how consistent the judgements in the matrix are, you can compute a consistency ratio that compares your consistency to that of a random set of judgements (hopefully you are more consistent than random judgements!). It turns out that in solving the general problem of extracting the relative weights from the paired comparison matrix, we can automatically generate a value related to consistency, the *eigenvalue*. We will be able to tell how *internally consistent* we are, not how 'real' we are. Everyone is familiar with this common human characteristic where we consistently *overestimate* or consistently *underestimate* in some situation, and in some cases, our colleagues are consistent even when knowing nothing about the topic!

The key value to assessing consistency in the AHP approach is called the maximum eigenvalue. It won't take much work to get it from our paired comparison matrix and once we have it, we have an additional way to improve judgements.

Calculating the Consistency Index and Consistency Ratio of the 'Weight' Matrix

MatrixWR • Priority vector = 3 * Priority vector (eigenvector)

$$\begin{pmatrix} 1 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \bullet \left\{\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right\} = \left\{\frac{12}{7}, \frac{6}{7}, \frac{3}{7}\right\}$$

$$\left\{\frac{12}{7}, \frac{6}{7}, \frac{3}{7}\right\} / \left\{\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right\} = \{3, 3, 3\}$$ "3" is the eigenvalue

Consistency index = (3 - 3)/2 = 0 *perfect!*

Consistency Ratio = 0/ 0.58 = *perfect!* (the 0.58 is from the random consistency index table below)

Note that "•" refers to matrix multiplication (matrix inner product)

FIGURE 7. Matrix of Paired Comparisons and Consistency Calculations

So, if I take the sum of these components and divide by 3, I of course get back 3. This is called the principle eigenvalue, (also called the 'characteristic value, which is what 'eigen' means anyway).

A Numerical Example of a Tasty Judgement

In this next example I would like to show a table of paired comparisons that are strictly personal judgements. Aside from the interesting application of the method, it also is an illustration of how I can assess my own consistency of judgments. The context is that I am a home gardener and grow various vegetables like peas, carrots, lettuce, spinach, and a smattering of others. In this case I am going to judge my relative taste preference of peas versus carrots versus lettuce. I will use a questionnaire format to capture the comparisons. (I have a template that might prove useful to you as well. See the "Analytic Hierarchy Process Questionnaire" on page 14).

After I gather these ratios, I will use the steps listed in "Manual Procedure to Calculate the Priority Vector and Eigenvalue" on page 9. As a check on these calculations I also used a math package *Mathematica* to calculate the priority vector as well as the consistency ratio. The differences are very small between the approximate method and the 'math package' method.

O.k., here is the questionnaire filled with my taste preference ratios.

Context for Paired Judgements: A Taste Preference for Selected Vegetables

Column I	Absolute 9	Very Strong 7	Strong 5	Weak 3	Equal 1	Weak 3	Strong 5	Very Strong 7	Absolute 9	Column II
peas			5							carrots
peas		7								lettuce
carrots				3						lettuce

The numbers convert my personal judgements to a ratio scale. The first one says that I strongly think peas taste better than carrots, I very strongly prefer peas to lettuce and, finally, I think carrots are a little bit tastier than (weakly tastier than) lettuce. For these comparisons I only needed three judgments as the reciprocals can be filled in automatically and the 'diagonal' values are 1's. So, look at the "Veggie" matrix of paired comparisons below:

Veggie Matrix	Priority Vector	Eigenvalue
$\begin{pmatrix} 1 & 5 & 7 \\ \frac{1}{5} & 1 & 3 \\ \frac{1}{7} & \frac{1}{3} & 1 \end{pmatrix}$	{0.730645, 0.188394, 0.0809612}	3.06489

Consistency Index = (3.065 -3)/2 = 0.033
Consistency Ratio = 0.033/0.58 = 0.06 (*this is a good ratio since it is less that 0.10*)

Note: I used the math package *Mathematica* to do these calculations

FIGURE 8. Family Vegetable Preference Paired Comparison Matrix and Analysis

Just to ensure that you can do these calculations, here are the steps for the matrix above:

Column totals = {47/35, 19/3, 11}

Dividing each element of a column by its column total yields:

$$\begin{pmatrix} \frac{35}{47} & \frac{15}{19} & \frac{7}{11} \\ \frac{7}{47} & \frac{3}{19} & \frac{3}{11} \\ \frac{5}{47} & \frac{1}{19} & \frac{1}{11} \end{pmatrix}$$

Summing each row yields the un-normalized priority vector:

{2.17052,0.579558,0.249924}

Now sum up these values to get a total

Finally, divide each element of the un-normalized priority vector by this overall total to get the normalized priority vector:

0.723506, 0.193186, 0.0833079

*Notice that this priority vector approximation procedure is very close to the math package priority vector solution.

So, peas came out on top, with a preference ratio of 0.73/0.19=3.7 of peas to carrots and a ratio of 0.73/0.08 = 8.68 of peas to lettuce.

Notice that here is where the inconsistency is revealed. I said, in the paired comparison matrix, that I referred peas to carrots in the ratio of 5 to 1 while I said I preferred peas to lettuce in the ratio of 7 to 1. However, I also said I preferred carrots to lettuce in the ratio of 3 to 1. If I had been consistent this ratio would have been

Peas/Carrots * Carrots/Lettuce = 5 * 3 = 15 to 1!

Fortunately the AHP method allows me to check such inconsistencies and redo comparisons that are way out of line. (Keep in mind that *inconsistency is inevitable* and it is a major strength of the AHP method to be able to detect unreasonable inconsistencies).

What are the Plausible Pair Weightings?

Well, suppose I had one stack of book copies that had 50 books in it while another stack had only 2. Would I use 50/2 = 25? Saaty says no, and brings in a lot of psychological factors that suggest that we are best able to directly compare things in a range of about 1 to 10. Beyond this range, he suggests that another level of groupings are called for, roughly in orders of magnitude. (Engineering and physics have long described relative sizes, times, or other physical quantities in ranges of powers of 10, that is, orders of magnitude).

Here is his suggested range of weightings:

Priority Scale for AHP

The judgement scale below is from Saaty, 1980.The numbers 2, 4, 6, and 8 are available for intermediate judgements.

GIven Elements A & B, judge their relative importance/weight as below, read off the equivalent number and insert that number at: Use even numbers for intermediate discriminations Note that these numbers represent *ratios, so the '3' means A is three times as weighty as B* Row A, Column B	Numeric Value
If A and B are equally important/weighty: insert -->	1
If A is weakly more important/weighty that B: insert -->	3
If A is strongly more important/weighty than B: insert -->	5
If A is very strongly more important/weighty than B: insert -->	7
If A is absolutely (extremely) more important/weighty than B: insert -->	9

Random Consistency Index Table

Saaty calculated hundreds of random matrices (with reciprocals forced) to get a baseline consistency index. Here are the random consistency indices: These are the numbers that will be the denominators of *Consistency Ratio* calculations.

DImension of (Square) Comparison Matrix	Random Consistency Index
2	0
3	0.58
4	0.9
5	1.12
6	1.24
7	1.32
8	1.41
9	1.45
10	1.49

A Questionnaire Format For Getting the Paired Comparisons

Saaty suggests a plausible way to collect these paired comparisons from a questionnaire. The idea is to set up the questions so that the decision maker can conveniently judge A against B or judge B against A, with the intent of avoiding fractional primary judgments.

The template below is set up for four factors, although it could easily be used for three factors by deleting three lines or accommodate more judgements by adding lines.

Analytic Hierarchy Process Questionnaire

Keeping the *context* in mind, judge column I's importance relative to column II (coming in from left) that is, column I is in the numerator of the ratio and column II is the denominator. Alternatively, if more convenient, judge the importance of column II versus column I (coming in from the right) thus thinking of column II as the numerator and column I as denominator. The idea is to avoid your needing to make fractional judgments. Enter the closest numeric you can using 8, 6, 4, and 2 for even finer distinctions.

Context for Paired Judgements: ___________________________________

Column I	Absolute 9	Very Strong 7	Strong 5	Weak 3	Equal 1	Weak 3	Strong 5	Very Strong 7	Absolute 9	Column II
C_1										C_2
C_1										C_3
C_1										C_4
C_2										C_3
C_2										C_4
C_3										C_4

A General Insight into the AHP Process

When you look at example hierarchies later in this manual, or when you start to construct your own hierarchies, you might keep in mind the following guidelines for understanding what is going on.

If I start from the top of a decision hierarchy and go "down" by calculating lower level priority vectors against higher level criteria, I am *analyzing*, or *reducing* the task to manageable chunks, that is, priority vectors at each level. When I reach the bottom of the hierarchy I have a collection of priority vectors reaching up through multiple levels. The question now is how to combine these vectors to show the impact of the bottom level on the very top.

The answer is to *synthesize* "up" the hierarchy all of the intermediate priority vectors into a *final overall priority vector*. To do this synthesis process, I combine each level's priorities into a higher level priority vector, weighted by the priority vector of the level above. I do this until I again reach the top and have an "overall" priority vector that reflects all of the intermediate level priority vectors.

An example of this dual process is shown next in symbolic form, and then I do the same example using numbers. If you would be more comfortable starting with the numeric analysis/synthesis, then start at "A Numeric Analysis of the Document Management System Purchase Decision" on page 16.

A Symbolic Analysis/Synthesis of a Document Management System Purchase Decision

Suppose you have to choose and justify purchasing/leasing some type of equipment, whether its a car, truck, milling machine, computer, or maybe a training DVD program. I have chosen to consider purchasing a Document Management System from one of possibly three vendors who offer systems: X, Y, Z, respectively. To be a little more systematic, I will introduce three criteria to use in judging the best system. My decision hierarchy is shown next.

The hierarchy below shows a 3-level skeleton decision structure that has the overall objective on top, which is the purchase of a Document Management System, abbreviated as "D". I have set up three criteria at level two: Total Cost, Technical Component, and Maintenance/Service. At level 3 I have set up the three alternatives or choices of Brand X, Y, or Z.(The method works best when all of these decision features are independent, or nearly so.)

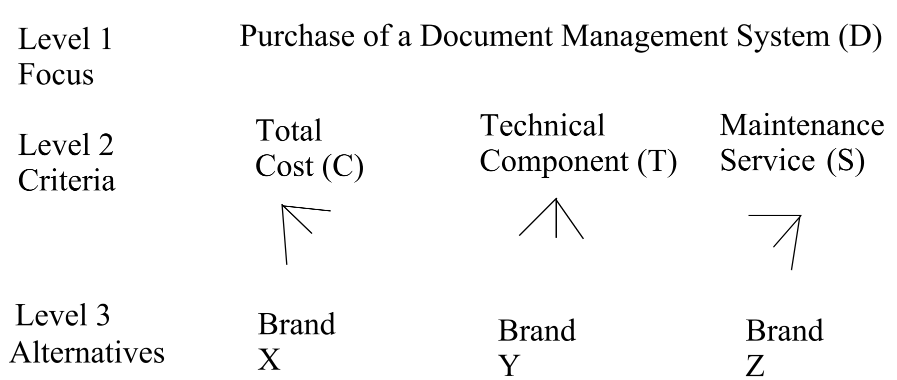


FIGURE 9. Three Level Hierarchy for Capital Acquisition

Appreciating the Analysis / Synthesis Processes Within AHP

Let me take the hierarchy above and show how I would find the priorities of the three brands X, Y, Z relative to the top goal, “D”, by taking into account intermediate priority vectors associated with level 2, “Criteria”. So, here is the way I would approach this task, going from top to bottom, computing priority vectors as I go:

1. Analysis: in the context of D: do pairwise comparisons, compute the priority vector of C, T, and S
2. Analysis: in the context of C: do pairwise comparisons, compute the priority vector of X, Y, Z
3. Analysis: in the context of T: do pairwise comparisons, compute the priority vector of X, Y, Z
4. Analysis: in the context of S: do pairwise comparisons, compute the priority vector of X, Y, Z
5. Synthesis: Weight the three priority X, Y, Z vectors with the C, T, S priority vector. The result is an overall priority X, Y, Z vector that synthesizes all the individual priority vectors and is now relative to “D”.

Note that the numbers below, such as **(1)**, **(2)**, and so on, are keyed to the numbered list above.

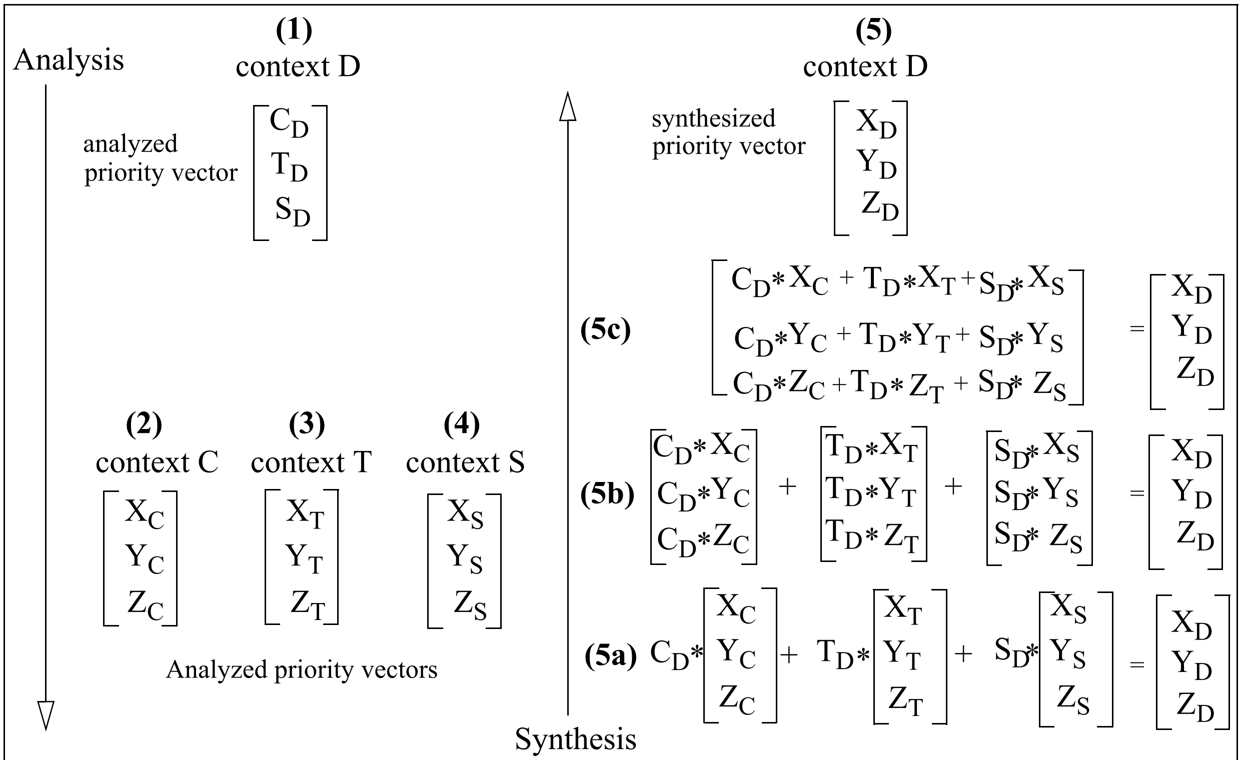


FIGURE 10. Symbolic Analysis / Synthesis Process of the DMS Purchase Decision

Let me point out that the objective of finding the priorities of X, Y, Z in the context of “D” was achieved via the intermediate criteria C, T, and S.

Let me now do the same analysis using actual numbers which some readers will prefer to the abstract discussion above.

A Numeric Analysis of the Document Management System Purchase Decision

I did this procedure above just using symbolic vectors. Now I will fill out the tables with actual numbers and compute priority vectors and eigenvalues.

Finding the Level 2 Priority Vector (in the context "D")

TABLE 2. In the Context of Purchasing a Document Management System (D)

(Context of Level 1) Document Management System D	Total Cost (C)	Tech Component (T)	Reliable Maintenance Services (S)
Total Cost (C)	1	1/5	1/3
Tech Component (T)	5	1	4
Reliable Maintenance Services (S)	3	1/4	1

In the context of the Level 1 (D): I would judge the Tech Component is Strongly more important than Total Cost and so I give it a "5". Further, I think that Reliable Maintenance/Services is weakly more important than Total Cost and so a "3". Finally, I judge that Tech Component is slightly more than weakly important than Reliable Maintenance/Services, so I give it a "4". The rest of the matrix can be filled out automatically.

Level 2 priority vector: C_D, T_D, S_D = {0.10, 0.67, 0.22}

Eigenvalue = 3.09, CR = 0.07

Finding a Level 3 Priority Vector (in the context of Total Cost)

TABLE 3. Judging Brands in the Context of Total Cost

(Context) Total Cost (C)	Brand X	Brand Y	Brand Z
Brand X	1	4/6	4/5
Brand Y	6/4	1	6/5
Brand Z	5/4	5/6	1

Level 3 priority vector X_C, Y_C, Z_C= {0.27, 0.4, 0.33}

Eigenvalue = 3.0 (this is perfect since we had objective costs for X, Y, Z)

CR = 0.0

Finding a Level 3 Priority Vector (in the context of Tech Component)

TABLE 4. Judging Brands in the Context of Tech Component

(Context) Total Cost	Brand X	Brand Y	Brand Z
Brand X	1	3	1/2
Brand Y	1/3	1	1/7
Brand Z	2	7	1

Level 3 priority vector X_T, Y_T, Z_T = {0.29, 0.09, 0.61}

Eigenvector = 3.003, CR = 0.0015/0.58 = 0.002

Finding a Level 3 Priority Vector (in the context of Reliability/Services)

TABLE 5. Judging Brands in the Context of Reliable Maintenance/Services

(Context) Reliable Maintenance Services	Brand X	Brand Y	Brand Z
Brand X	1	1/5	1/3
Brand Y	5	1	4
Brand Z	3	1/4	1

Level 3 priority vector X_S, Y_S, Z_S = {0.10,0.67,0.23}

Eigenvalue = 3.09, CR = 0.07

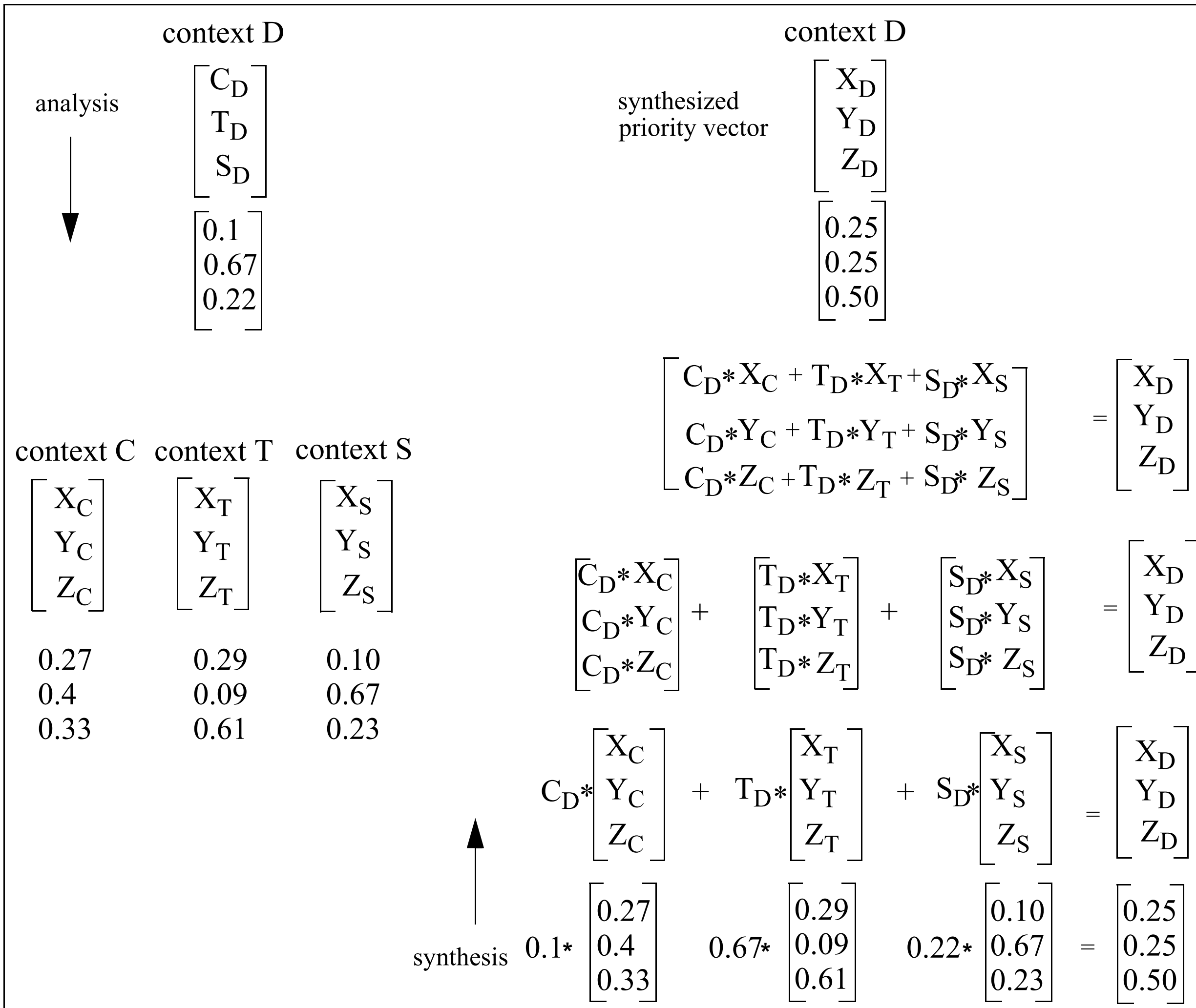


FIGURE 11. Analysis / Synthesis of Example B1 in AHP

From this result, rounded to two decimal places, it appears that brands B and C are the same, on the criteria used while brand A is twice a good/important/desirable as wither B or C.

Summary

This has been a very quick (very incomplete) exploration of some of Thomas Saaty's Analytic Hierarchy Process (AHP). From all reports, over the last 25 years, the method is most useful in the kinds of unstructured decision tasks in the context of the environment called *life*. It seems he has tapped into a common form of human thinking, that of hierarchies and their leveled inter-relationships. Using our own minds and experiences as the measuring rod for paired judgements brings just about all of the human determined decision tasks under this one approach. I have used this approach myself off and on for many years, introducing it to students in the mid 1980's. It works, or should I say, it works like humans work, with all our inconsistencies, uncertainties, and lack of knowledge. Still, decisions must be made and this AHP approach seems to be among the very few to provide some defensible guidance to the decisions we come up with, and then have to justify.

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************Supplemental Examples**********************

Example NB2: Numerical Calculations for Example B2 (with nested criteria)

Here is the hierarchy diagram again with some numbers I would likely use in such a decision. There is one calculation procedure to note here. Level 3 consists of nested criteria, F1, F2, and F3 and so those will generate a priority vector relative to the Tech Component. Then, F1, F2, and F3 will provide the context for finally judging X, Y, and Z relative to the Tech Component.

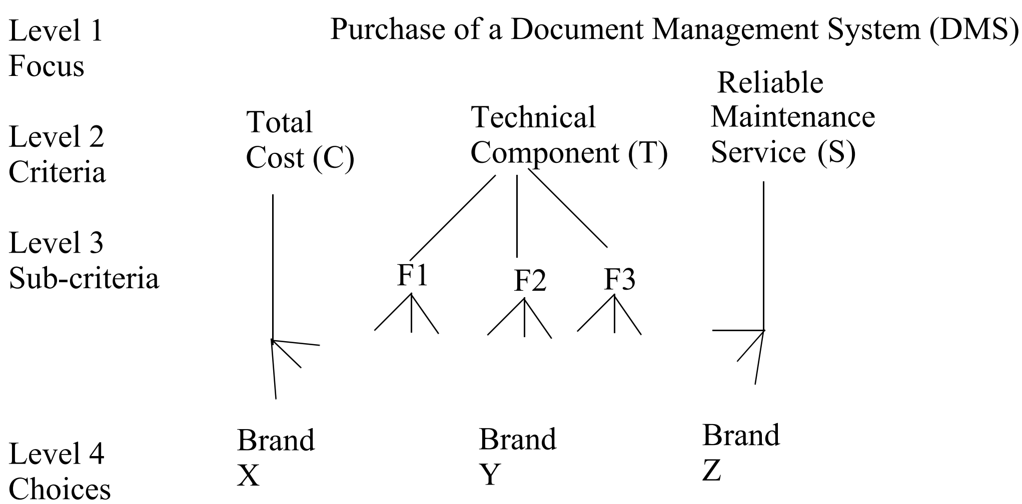

FIGURE 12. Four Level Hierarchy for Hardware/Software Acquisition

Starting from the top down, The Priorities of Level 2 in the Context of Level 1.

In the context of the final Level 1 Focus: I would judge that the Tech Component is Strongly more important than Total Cost and so I give it a “5”. Further, I think that Reliable Maintenance Services is weakly more important than Total Cost and so a “3”. Finally, I judge that Tech Component is slightly more weakly important than Reliable Maintenance/Services, so I give it a “4”.

TABLE 6. Judging in the Context of Purchasing a Document Management System

(Context) Document Management System	Total Cost	Tech Component	Reliable Maintenance Services
Total Cost	1	1/5	1/3
Tech Component	5	1	4
Reliable Maintenance Services	3	1/4	1
			1

The priority vector and eigenvalue for Level 2 in the context of the overall Level 1 purchase goal:

Eigenvalue = 3.09

Level 2 Priority Vector = {0.1, 0.67,0.22}

This shows Tech Component three times as important as Reliability and strongly+ more important than Total Cost.

Level 3 Priority Vectors, Nested Within the Technical Component

For Level 3 things are a bit different since Level 3 consists only of technical features *within* the Tech Component. The context here is the overall idea of the Technical Component and the relative importance of its sub features. To start with: I think that F1 is weakly more important than F2, and F3 is strongly more important than F1 and that F3 is very strongly more important than F2.

TABLE 7. Judging Tech Features within the Tech Component

(Context) Tech Component	F1 (XML/XSLT pipeline)	F2 (SVG/ MathML)	F3 (Relational/ XML DB)
F1 (XML /XSLT pipe-line)	1	3	1/5
F2 (SVG/MathML)	1/3	1	1/7
F3 (Relational/XML DB)	5	7	1

Priority Vector ={0.19, 0.081, 0.73}

Eigenvalue = 3.06

Consistency Index (CI) = (3.06-3.00)/2 = 0.03

Consistency Ratio = 0.03/0.58 = 0.05 (where the 0.58 is from the "Random Consistency Index Table" on page 12

This is a good ratio since it is much less than the critical value of "0.1"

Level 4: Judging Brands in the context of Total Cost

Here is a case where I can actually use objective data, since the prices are in comparable dollars. Suppose Brand X is $40,000, Brand Y is $60,000, while Brand Z is $50, 000. Now I can calculate actual ratios without needing my personal judgements. For example, the ratio of Brand X / Brand Y can be calculated as:

40000/50000 = 4/5 = 0.8, so that's what goes in the matrix. Since these are absolute values, then the consistency should be perfect, as it is. The priority vector could also be calculated without any work just by looking at the original values. This is so since the total is 150,000 and we have 40,000/ 150000, 60000/150000, 50000/150000. So I can write the priority vector as:

Priority Vector = {4/15, 6/15, 5/15}

Consistency Ratio = 0 * (this matrix is perfectly consistent!)

TABLE 8. Judging Brands in the Context of Total Cost

(Context) Total Cost	Brand X	Brand Y	Brand Z
Brand X	1	4/6	4/5
Brand Y	6/4	1	6/5
Brand Z	5/4	5/6	1
			1

Level 4: Judging Brands in the Context of the Three Sub-Tech Features

TABLE 9. Judging Brands in the Context XML/XSLT Pipeline (F1)

(Context) XML/XSLT Pipeline	Brand X	Brand Y	Brand Z
Brand X	1	1/5	1/3
Brand Y	5	1	4
Brand Z	3	1/4	1
			1

TABLE 10. Judging Brands in the Context SVG/MathML (F2)

(Context) SVG/MathML	Brand X	Brand Y	Brand Z
Brand X	1	1/5	1/3
Brand Y	5	1	4
Brand Z	3	1/4	1
			1

TABLE 11. Judging Brands in the Context Relational/XML DB

(Context) Relational/XML DB	Brand X	Brand Y	Brand Z
Brand X	1	1/5	1/3
Brand Y	5	1	4
Brand Z	3	1/4	1
			1

Level 4: Judging Brands in the Context of Reliability/Services

TABLE 12. Judging Brands in the Context of Reliable Maintenance/Services

(Context) Reliable Maintenance Services	Brand X	Brand Y	Brand Z
Brand X	1	1/5	1/3
Brand Y	5	1	4
Brand Z	3	1/4	1
			1

**************************EXTRA VERBIAGE ******************

Now, since I have the actual weights I can calculate what the paired comparison ratios would be.

Using the scale, I measure the weight of stack A and the weight of stack B, I get 4 lb. for A and 2 lb. for B. The ratio is 4/2. (Without the scale, I would hold stack A in one hand and stack B in the other and judge the ratio of weights, recording a value for this ratio in the table below ("Weight Ratios of Stacks of Books - A Table of Paired Comparisons" on page 7.)

Since I measured the weight of stack A to the weight of stack C, I get 4 lb. for A and 1 lb. for C, yielding a ratio of 4/1. (Without the scale I would have held stack A in one hand and stack C in the other and recorded a value for that judged ratio in the table.)

Finally, I compare the weight of stack B to stack C and get a ratio of 2/1.

Suggestive Hierarchy Examples

I would like to describe some of the uses of hierarchies as drawn from the literature and especially from Saaty's work. I have reworked all of them to be of immediate use to students and possibly to other colleagues teaching the AHP ideas. Note that many more factors can be introduced at each level of the hierarchies presented below. Finding and organizing these factors is, of course, the hardest part. The next hardest is getting busy people to take the time to carefully consider their assessments of one factor against another. The math needed to analyze these decisions, while not easy, is standard and a number of packages are available to help here.

These hierarchies are broken down into common categories such as:

- Business/Organizational priorities - capital equipment choices, financial decisions
- Jobs and Personnel decisions - assessing job importance, skills set analyses, personnel evaluations, student evaluations, faculty evaluations,
- Personal decisions - course evaluations, buying a car, a house, attending a school
- Planning and economic policies

It's inevitable that your product or service will grow obsolete from either external factors such as competition, business/technical advances, or your own internal structure (such as the aging of employees/machines associated with the product/service). The context below assumes that the responses are limited to consideration of how to handle existing products or perhaps enter a new market. Various responses to this recognition are open including deleting product A and emphasizing product B or product C, or continuing reduced support of A. Of course many more factors could be included here, this is just a start.

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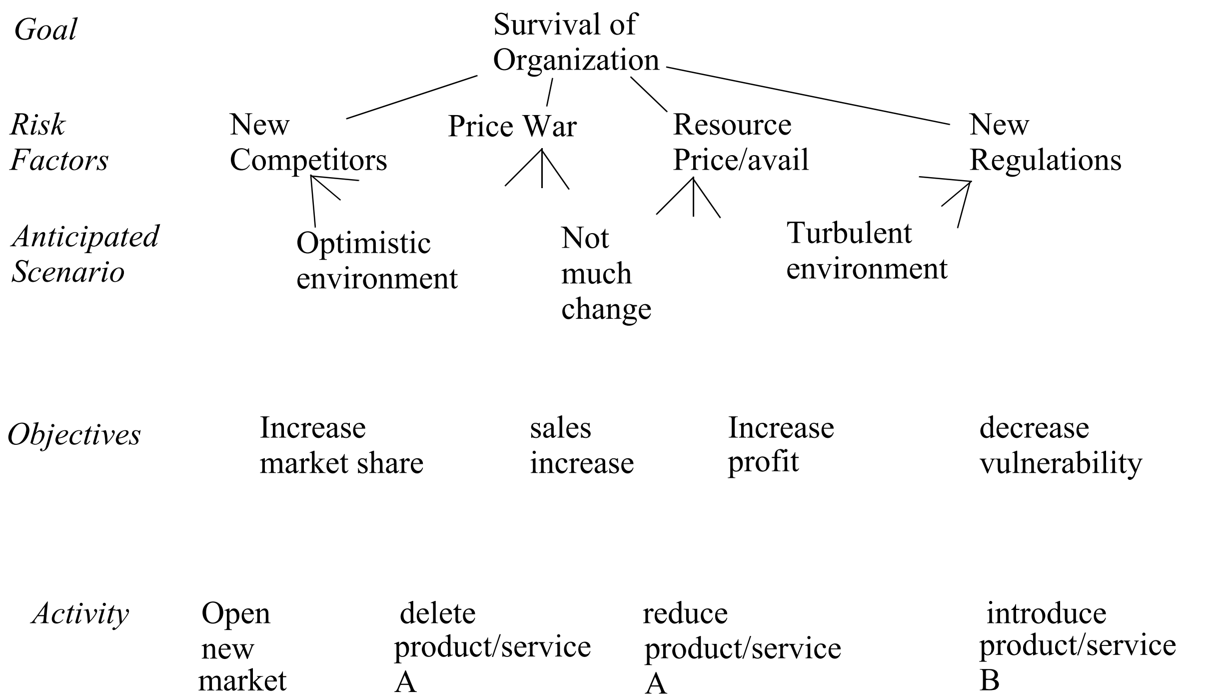


FIGURE 13. Survival of an Organization Depending on Product/Service Activities

*************WRONG IDEA, NEED ALTERNATIVES!!!!**

Example 1: Select A Romance Novel

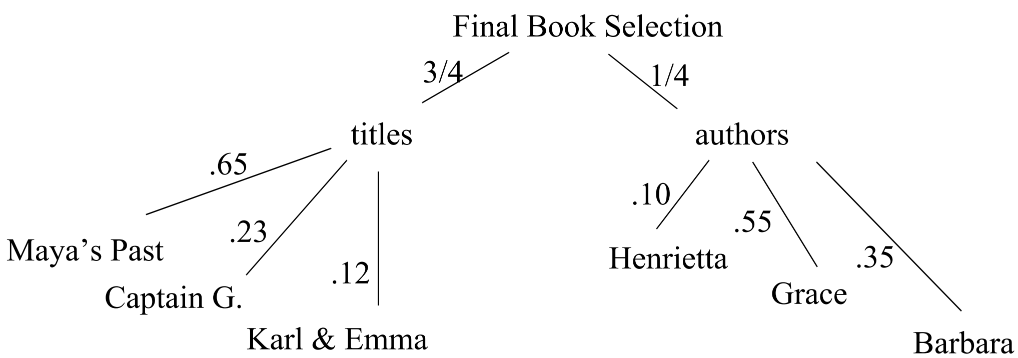
Say I am going online to order a romance novel for my reading pleasure. Unfortunately, the specialized site I deal with is experiencing technical difficulties (surprise!), and can only show me three choices and, minimal information about each. I are determined though to order a novel, but which one? I can see the title, author, and price, but that's it. I'm a little bit conflicted here since the topics are not all equally appealing, given the mood I'm in, and, I also know a little about the three authors listed and have some preferences among them, not overwhelming, but significant. Somehow I would like to take into account both the importance of a title to me as well as the importance of an author to me. What I mean is that right now, I am going to give more weight to how the title sounds to me rather than who the author is. I will quantify that judgement by saying that titles are 3 times more important to me than is an author. (This is independent of the actual title and actual author). Price is a not a factor for now, so that's just background information.

By applying a standard procedure called the AHP process, I will assess the final preferences for each novel, taking into account an *intermediate* layer of judgments involving titles and authors. As a bonus, we can also calculate an overall measure of consistency! While not too important in this case, this built in capability of the method will be of considerable value when confronting critical, complicated issues. So, the result is that we will have a ranking of the three books based on their rankings within title and author plus a measure of our higher level ranking of titles to authors within the top level goal of selecting a book. Ok, here are the books I want to choose from:

Romance Novels Available (Note: I made all this up!)

- *Maya's Past Returns*, by Henrietta Stackpool, $24.95
- *Captain Garson, Bucaneer Extraordinaire*, by Grace Metalous, $20.50
- *Karl and Emma Adrift*, by Barbara Kinsolving, $19.95

Ok., what I do is to start with the two high level goals, titles, and authors. Of these two, which is more important, in the context of the top most goal? I'll use a scale of 1-9 here where 1 means same importance and a 9 indicates an overwhelming importance. Intermediate numbers suggest intermediate preference intensities. A 3, in this case, translates to saying one factor/variable is weakly more important than another. These preferences are directional, since in this case I am comparing titles against authors. In my case, I weakly prefer titles to authors. This means that when I read the title, that is 3 times more important to me than the author. (The scale is shown in the section "Priority Scale for AHP" on page 12.)



My Final Book Choice

Maya's Past		0.51
Captain G	=	0.31
Karl & Emma		0.18

Final Choice Context	titles	authors
titles	1	3
authors	1/3	1

$$3/4 * \begin{matrix} 0.65 \\ 0.23 \\ 0.12 \end{matrix} + 1/4 * \begin{matrix} 0.10 \\ 0.55 \\ 0.35 \end{matrix} = \begin{matrix} 0.51 \\ 0.31 \\ 0.18 \end{matrix}$$

Final Choice Priority Vector

3/4
1/4

Titles Context	Maya	Captain G	K&E
Maya's Past	1	3	5
Captain G	1/3	1	2
Karl & Emma	1/5	1/2	1

Authors Context	Henrietta	Grace	Barbara
Henrietta	1	1/3	1/7
Grace	3	1	3
Barbara	7	1/3	1

Title Context Priority Vector

0.65
0.23
0.12

Consistency

Authors Context Priority Vector

0.10
0.55
0.35

Consistency

FIGURE 14. Romance Novel Choice Priorities

The final decision is based on the calculated priorities that show M*aya's Past Returns* as the preferred book by a ratio of 51/31 to the *Captain Garson* book and a ratio of 51/18 over the *Karl and Emma Adrift* novel.